iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

The induction coefficient of induced representation theory: its algebra and an analytic expression

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1987 J. Phys. A: Math. Gen. 20 1065

(http://iopscience.iop.org/0305-4470/20/5/017)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 05:24

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **20** (1987) 1065-1074. Printed in the UK

The induction coefficient of induced representation theory: its algebra and an analytic expression

R W Haase† and R Dirl

Institut für Theoretische Physik, Technische Universität Wien, A-1040 Karlsplatz, 13 Wien, Austria

Received 25 July 1985

Abstract. The induction coefficient is defined as the element of a matrix reducing the induced representation into irreducible constituents. We give the algebra of these coefficients and a closed analytic expression in terms of matrix elements of coset representatives. The consequences of this choice are derived and the importance of some of these results for the Racah-Wigner coupling algebra of the symmetric and unitary groups noted.

1. Introduction

The method of induced representations in physical applications has been both useful and powerful. For finite groups the method originates from Frobenius (1898). This method was extended to continuous groups by Wigner (1939) in classifying the unitary irreps of the Poincaré group. In more recent applications the induction method and its related concept of coset manifolds are fundamental to the Kaluza-Klein theories of supergravity in elementary particle physics (Salam and Strathdee 1982, van Nieuwenhuizen 1984) and to the definition of coherent state theory of, for example, $Sp_6 \supset U_3$ which is used to describe collective nuclear behaviour (Perelomov 1975, Monastryrskii and Perelomov 1975, Kramer 1982, Rowe 1984). In the field of molecular and crystal field theory, induced representations of the symmetric groups have been employed to classify the many-electron states of multi-centred systems (Chan and Newman 1984).

The systematic analysis of the general properties of induced representations was performed by Mackey (1951, 1952, 1953, 1968) (see Barut and Rączka 1977). One important task is to evaluate the equivalence matrix (the elements of which are called induction coefficients by Haase and Butler (1984a)) which reduces the induced representation into irreducible representations (irreps). Two main approaches have been given. The first is the projection operator techniques with the regular representation as its starting point. Sullivan (1973, 1975) has developed this analysis further by concentrating on the double coset matrix elements (DCME) and 'weighted' DCME and giving their properties with respect to induced representations. Haase and Butler (1984a, 1985) has given a second perhaps more direct approach defining instead certain transformation coefficients—induction coefficient, induction and reinduction factors—and developing the algebra which they satisfy. The two sets of transformation coefficients from these two approaches are quite different yet they perform identical

† Present address: Institute of Theoretical Physics, Universität Tübingen, D-4700 Tübingen 1, West Germany.

0305-4470/87/051065+10$02.50 © 1987 IOP Publishing Ltd

transformations. We reconcile the two approaches by identifying the induction coefficient with a weighted coset matrix element (weighted CME). The identification is based on the Frobenius reciprocity theorem which underlies many of the results in induced representation theory.

We have organised this paper in the following way. In § 2 we briefly give some of the properties of the induction coefficients, induction factors and reinduction factors and their interrelationships. The new results here are the factorisation of the induction coefficient and the expansion of the reinduction factor in terms of four induction factors and a second reinduction factor. An erratum is also given for Haase and Butler (1984a, equation (5.7)). In § 3 we show that the weighted CME choice for the induction coefficient fixes the entire algebra of the induced representation theory. It also determines the induction and reinduction factors to be equal to Sullivan's weighted DCME and a resubduction factor. An application to $SO(n+1) \supset SO(n)$ induced representations is given. Throughout the paper we use the notation and terminology established in Haase and Butler (1984a). For all mathematical derivation we use implicitly the concept of a G basis for which the matrix representations of distinct equivalent representations are taken to be identical. That is, in any one basis we choose a representation and fix it once and for all.

2. Induced representation theory

In this section we briefly recount definitions of three types of transformation coefficients describing changes of bases within induced representation spaces. The details can be found elsewhere (Haase and Butler 1984a). The three types of coefficients were therein termed induction coefficients, induction factors and reinduction factors. Later in this section we describe the relationship between these three. For simplicity we shall assume in the following that the groups are finite or compact continuous and if $G \supset H$ then H is of finite index in G or G/H is compact.

The product space of irrep space $V_{y\eta}$ of a subgroup $H \subset G$ with the left coset space $V_{G/H}$ defines the induced representation space

$$V_{y\eta(H)\uparrow G} = V_{G/H} \otimes V_{y\eta}. \tag{2.1}$$

This is in general reducible

$$V_{y\eta(H)\uparrow G} = \bigoplus_{a\gamma} V_{y\eta(H)\uparrow a\gamma(G)} \tag{2.2}$$

where γ labels the irreps of G and $a = 1, 2, \ldots |\eta\uparrow:\gamma|$ indexing the $|\eta\uparrow:\gamma|$ multiple occurrences of $\gamma(G)$ in $\eta(H)\uparrow G$. We note for convenience that the Frobenius reciprocity theorem gives the result

$$|\eta\uparrow:\gamma| = |\gamma:\eta| \tag{2.3}$$

where $|\gamma:\eta|$ denotes the multiplicity of $\eta(H)$ in $\gamma(G)$ under $G \supset H$. The reduction (2.2) is given by the induction coefficients. Let p denote a coset representative of G/H, and i and j label the bases of irrep spaces of γ and η respectively. Then the *induction coefficient* $\langle\eta(H)\uparrow a\gamma(G)i|\eta(H)\uparrow Gpj\rangle$ is an element of an orthonormal matrix indexed by sets $(a\gamma i)$ and (pj). In terms of matrix representations it performs the reduction

$$\sum_{p'j'pj} \langle\eta(H)\uparrow a'\gamma'(G)i'|\eta(H)\uparrow Gp'j'\rangle \eta\uparrow(g)^{p'j'}{}_{pj} \langle\eta(H)\uparrow Gpj|\eta(H)\uparrow a\gamma(G)i\rangle = \delta^{a'}{}_{a}\delta^{\gamma'}{}_{\gamma}\gamma(g)^{i'}{}_{i} \tag{2.4}$$

where $\eta\uparrow(g)^{p'j'}{}_{pj}=\delta^{p'}{}_{p_0}\,\eta(p_0^{-1}gp)^{j'}{}_j$ with p_0 satisfying $p_0^{-1}gp\in \mathrm{H}$. The *induction factor* is an element of the matrix transforming between representations equivalent by Mackay's subgroup theorem:

$$[\eta(\mathrm{H})\uparrow \mathrm{G}]\downarrow K\simeq\bigoplus_q[\eta(\mathrm{H}_q)\downarrow \mathrm{L}(q)]\uparrow \mathrm{K} \tag{2.5}$$

where q denotes a double coset representative of $\mathrm{H}\backslash\mathrm{G}/\mathrm{K}$, $\mathrm{H}_q\equiv q\mathrm{H}q^{-1}$ and $L(q)=\mathrm{H}_q\cap \mathrm{K}$. We denote the induction factor as

$$\langle\eta(\mathrm{H})\uparrow a\gamma(\mathrm{G})b\kappa(\mathrm{K})|[\eta(\mathrm{H}_q)c\lambda(\mathrm{L}(q))]\uparrow d\kappa(\mathrm{K})\rangle. \tag{2.6}$$

These factors are elements of an orthonormal matrix indexed by $(a\gamma b)$ and $(c\lambda(L(q))d)$. We note that this includes all groups $\mathrm{L}(q)$ indexed by q. In the case $\mathrm{K}=\mathrm{E}$ (the identity group) the induction factor reduces to an induction coefficient.

The induction factor can also be used to factor an induction coefficient of $\eta(\mathrm{H})\uparrow \mathrm{G}$. Each coset representative $p\in \mathrm{G}/\mathrm{H}$ can be uniquely expressed as $p=rq$ where $q\in \mathrm{K}\backslash\mathrm{G}/\mathrm{H}$ and r is a coset representative of $\mathrm{K}/\mathrm{L}(q)$ (see Coleman 1966, Bradley and Cracknell 1972). By performing appropriate bases transformations we derive

$$\begin{aligned}
&\langle\eta(\mathrm{H})\uparrow a\gamma(\mathrm{G})i|\eta(\mathrm{H})\uparrow \mathrm{G}pj\rangle\\
&\quad=\sum_{b\chi kc\lambda ld}\langle\gamma(\mathrm{G})i|\gamma(\mathrm{G})b\kappa(\mathrm{K})k\rangle\\
&\qquad\times\langle\eta(\mathrm{H})\uparrow a\gamma(\mathrm{G})b\kappa(\mathrm{K})|[\eta(\mathrm{H}_q)c\lambda(\mathrm{L}(q))]\uparrow d\kappa(\mathrm{K})\rangle\\
&\qquad\times\langle\lambda(\mathrm{L}(q))\uparrow d\kappa(\mathrm{K})k|\lambda(\mathrm{L}(q))\uparrow \mathrm{K}rl\rangle\\
&\qquad\times\langle\eta(\mathrm{H})c\lambda(\mathrm{L}(q)_{q^{-1}})l|\eta(\mathrm{H})j\rangle.
\end{aligned} \tag{2.7}$$

The fourth term is a second induction coefficient of a lower chain than $\mathrm{G}\supset\mathrm{H}$, that of $\mathrm{K}\supset\mathrm{L}(q)$. The second and fifth terms are transformation coefficients reducing respectively the irrep spaces $\gamma(\mathrm{G})$ and $\eta(\mathrm{H})$ according to the GH chain $\mathrm{G}\supset\mathrm{K}$ and $\mathrm{H}\supset\mathrm{L}(q)_{q^{-1}}$. Using their orthonormality property we have

$$\begin{aligned}
&\langle\eta(\mathrm{H})\uparrow a\gamma(\mathrm{G})b\kappa(\mathrm{K})k|\eta(\mathrm{H})\uparrow \mathrm{G}pc\lambda(\mathrm{L}(q)_{q^{-1}})l\rangle\\
&\quad=\sum_d\langle\eta(\mathrm{H})\uparrow a\gamma(\mathrm{G})b\kappa(\mathrm{K})|[\eta(\mathrm{H}_q)c\lambda(\mathrm{L}(q))]\uparrow d\kappa(\mathrm{K})\rangle\\
&\qquad\times\langle\lambda(\mathrm{L}(q))\uparrow d\kappa(\mathrm{K})k|\lambda(\mathrm{L}(q))\uparrow \mathrm{K}rl\rangle.
\end{aligned} \tag{2.7a}$$

Equation (2.7*a*) can be likened to Racah's factorisation lemma for coupling coefficients (see Butler 1975).

The *reinduction factor*,

$$\langle\lambda(\mathrm{L})\uparrow a\kappa(\mathrm{K})\uparrow d\gamma(\mathrm{G})|\lambda(\mathrm{L})\uparrow a\eta(\mathrm{H})\uparrow b\lambda(\mathrm{G})\rangle \tag{2.8}$$

is an element of an orthonormal matrix indexed by $(c\kappa d)$ and $(a\eta b)$ and describing the equivalence of the two induced representations

$$(\lambda(\mathrm{L})\uparrow \mathrm{K})\uparrow \mathrm{G}\simeq(\lambda(\mathrm{L})\uparrow \mathrm{H})\uparrow \mathrm{G}. \tag{2.9}$$

Occurring in (2.9) is a two-step induction process from L to G via two intermediate groups H and K. At each step the reduction of the induced representation space to irreducibles has been performed. Since the reduction involves an induction coefficient, a reinduction factor can be expressed in terms of four induction coefficients. Each element of G can be expressed as

$$g=p_1h=p_1p_2l_1 \qquad \text{and} \qquad g=p_3k=p_3p_4l_2 \tag{2.10}$$

where $p_1 \in G/H$, $p_2 \in H/L$, $p_3 \in G/K$, $p_4 \in K/L$. But note that the elements l_1 and l_2 of L are not necessarily the same. However we may write $l_2 = l_0 l_1$. Thus we have the following

$$\sum_{a\eta bj} \langle \lambda(L) \uparrow c\kappa(K) \uparrow d\gamma(G) | \lambda(L) \uparrow a\eta(H) \uparrow b\gamma(G) \rangle$$
$$\times \langle \eta(H) \uparrow b\gamma(G) i | \eta(H) \uparrow G p_1 j \rangle \langle \lambda(L) \uparrow a\eta(H) j | \lambda(L) \uparrow H p_2 l \rangle$$
$$= \sum_{kl'} \langle \kappa(K) \uparrow d\gamma(G) i | k(K) \uparrow G p_3 k \rangle \langle \lambda(L) \uparrow c\kappa(K) k | \lambda(L) \uparrow K p_4 l' \rangle \lambda(l_0)^{l'}{}_{l} \tag{2.11}$$

where p_1, p_2, p_3, p_4 are such that $l_0 = p_4^{-1} p_3^{-1} p_1 p_2 \in L$. Note the appearance of the matrix element $\lambda(l_0)^{l'}{}_{l}$. Unfortunately an error appears in equations (5.6)–(5.7) of Haase and Butler (1984a). There l_0 was assumed the unit element which is not true in general.

One final result is to use the factorisation of the induction coefficients, equation (2.7), to express the induction factor in terms of four induction factors and a second reinduction factor of a 'lower' group–subgroup scheme. The group relationships are given in figure 1. If given $M \subset G$ then the coset representatives $p_1 \in G/H$ and $p_3 \in G/K$ can be chosen as

$$p_1 = r_1 q_1 \qquad \text{with } q_1 \in M\backslash G/H \tag{2.12}$$

$$p_3 = r_3 q_3 \qquad \text{with } q_3 \in M\backslash G/K \tag{2.13}$$

where r_1 and r_3 are coset representatives of $M/N(q_1)$ and $M/R(q_3)$ respectively. $N(q_1)$ and $R(q_3)$ are defined as the intersection groups $N(q_1) \equiv H_{q_1} \cap M$ and $R(q_3) \equiv K_{q_3} \cap M$. The isomorphic groups $N(q_1)_{q_1^{-1}}$ and $R(q_3)_{q_3^{-1}}$ are subgroups of H and K respectively. The coset representatives p_2 and p_4 can now also be expressed in similar form

$$p_2 = r_2 q_2 \qquad \text{with } q_2 \in N(q_1)_{q_1^{-1}} \backslash H/L \tag{2.14}$$

$$p_4 = r_4 q_4 \qquad \text{with } q_4 \in R(q_3)_{q_3^{-1}} \backslash K/L \tag{2.15}$$

where r_2 and r_4 are coset representatives of $N(q_1)_{q_1^{-1}}/S(q_1, q_2)$ and $R(q_3)_{q_3}/S(q_3, q_4)$. Here $S(q_1, q_2) \equiv L_{q_2} \cap N(q_1)_{q_1^{-1}}$ and $S(q_3, q_4) \equiv L_{q_4} \cap R(q_3)_{q_3^{-1}}$. The isomorphic groups

$$S(q_1, q_2)_{q_1} \equiv q_1 S(q_1, q_2) q_1^{-1} \qquad \text{and} \qquad S(q_3, q_4)_{q_3^{-1}} \equiv q_3 S(q_3, q_4) q_3^{-1}$$

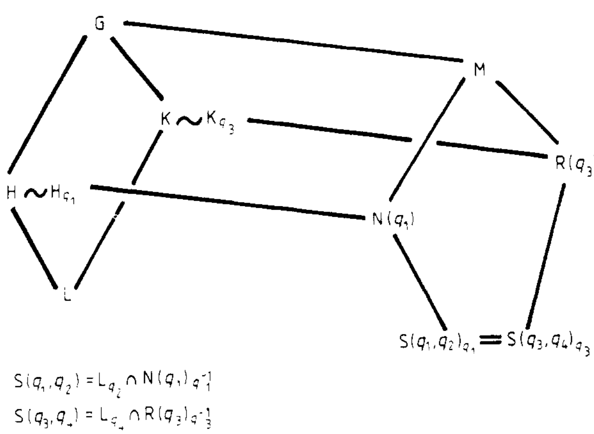

Figure 1. Group–subgroup relations for equations (2.17).

are subgroups of $\mathrm{N}(q_1)$ and $\mathrm{R}(q_3)$ respectively. Furthermore there exists for each pair (q_1, q_2) a unique pair (q_3, q_4) such that $\mathrm{S}(q_1, q_2)_{q_1} = \mathrm{S}(q_3, q_4)_{q_3}$. From (2.11) we have $l = p_4^{-1} p_3^{-1} p_1 p_2 \in \mathrm{L}$. The coset factorisation $p_i = r_i q_i$ gives

$$\begin{aligned} l_0 &\equiv q_4^{-1} r_4^{-1} q_3^{-1} r_3^{-1} r_1 q_1 r_2 q_2 \\ &\equiv (q_4^{-1} q_3^{-1}) s_0 (q_1 q_2) \end{aligned} \tag{2.16}$$

with $s_0 \equiv (q_3^{-1} r_4^{-1} q_3)(r_3^{-1})(r_1)(q_1 r_2 q_1^{-1})$. For the pairs (q_1, q_2) and (q_3, q_4) satisfying $\mathrm{S}(q_1, q_2)_{q_1} = \mathrm{S}(q_3, q_4)_{q_3} \equiv \mathrm{S}$ we have $s_0 \in \mathrm{S}$.

With the above group–subgroup relations specified, we substitute for each induction coefficient in (2.11) the corresponding factorisation (2.7) arriving at

$$\begin{aligned} \sum_{aa'\eta\nu bb'a_2} & \langle \lambda(\mathrm{L}) \uparrow c\kappa(\mathrm{K}) \uparrow d\gamma(\mathrm{G}) | \lambda(\mathrm{L}) \uparrow a\eta(\mathrm{H}) \uparrow b\gamma(\mathrm{G}) \rangle \\ & \times \langle \eta(\mathrm{H}) \uparrow b\gamma(\mathrm{G}) a_1 \mu(\mathrm{M}) | [\eta(\mathrm{H}_{q_1}) a_2 \nu(\mathrm{N}(q_1))] \uparrow b'\mu(\mathrm{M}) \rangle \\ & \times \langle \nu(\mathrm{N}(q_1)) \uparrow b'\mu(\mathrm{M}) m | \nu(\mathrm{N}(q_1)) \uparrow \mathrm{M} r_1 n \rangle \\ & \times \langle \lambda(\mathrm{L}) \uparrow a\eta(\mathrm{H}) a_2 \nu(\mathrm{N}(q_1)_{q_1^{-1}}) | [\lambda(\mathrm{L}_{q_2}) a_4' \sigma'(\mathrm{S}(q_1, q_2))] \uparrow a'\nu(\mathrm{N}(q_1)_{q_1^{-1}}) \rangle \\ & \times \langle \sigma'(\mathrm{S}(q_1, q_2)) a'\nu(\mathrm{N}(q_1)_{q_1^{-1}}) n | \sigma'(\mathrm{S}(q_1, q_2)) \uparrow \mathrm{N}(q_1)_{q_1^{-1}} r_2 s' \rangle \\ = \sum_{\varepsilon d\rho\sigma a_3 a_4 r} & \langle \kappa(\mathrm{K}) \uparrow d\gamma(\mathrm{G}) a_1 \mu(m) | [\kappa(\mathrm{K}_{q_3}) a_3 \rho(\mathrm{R}(q_3))] \uparrow d'\mu(\mathrm{M}) \rangle \\ & \times \langle \rho(\mathrm{R}(q_3)) \uparrow d'\mu(\mathrm{M}) m | \rho(\mathrm{R}(q_3)) \uparrow \mathrm{M} r_3 r \rangle \\ & \times \langle \lambda(\mathrm{L}) \uparrow c\kappa(\mathrm{K}) \uparrow a_3 \rho(\mathrm{R}(q_3)_{q_3^{-1}}) | [(\mathrm{L}_{q_4}) a_4 \sigma(\mathrm{S}(q_3 q_4))] \uparrow c'\rho(\mathrm{R}(q_3)_{q_3^{-1}}) \rangle \\ & \times \langle \sigma(\mathrm{S}(q_3 q_4)) \uparrow c'\rho(\mathrm{R}(q_3)_{q_3^{-1}}) r | \sigma(\mathrm{S}(q_3 q_4)) \uparrow \mathrm{R}(q_3)_{q_3^{-1}} r_4 s \rangle \\ & \times \langle \lambda(\mathrm{L}) a_4 \sigma(\mathrm{S}(q_3 q_4)_{q_4^{-1}}) s | l_0 | \lambda(\mathrm{L}) a_4' \sigma'(\mathrm{S}(q_1 q_2)_{q_2^{-1}}) s' \rangle . \end{aligned} \tag{2.17}$$

The expansion of l_0 in (2.16) is used to simplify the last term

$$\begin{aligned} & \langle \lambda(\mathrm{L}) a_4 \sigma(\mathrm{S}(q_3 q_4)_{q_4^{-1}}) s | l_0 | \lambda(\mathrm{L}) a_4' \sigma'(\mathrm{S}(q_1 q_2)_{q_2^{-1}}) s' \rangle \\ & \quad = \langle \lambda(\mathrm{L}_{q_3 q_4}) a_4 \sigma(\mathrm{S}(q_3 q_4)_{q_3}) s | s_0 | \lambda(\mathrm{L}_{q_1 q_2}) a_4' \sigma'(\mathrm{S}(q_1 q_2)_{q_1}) s' \rangle \\ & \quad = \langle \lambda(\mathrm{L}_{q_3 q_4}) a_4 \sigma(\mathrm{S}(q_3 q_4)_{q_3}) | \lambda(\mathrm{L}_{q_1 q_2}) a_4' \sigma(\mathrm{S}(q_1 q_2)_{q_1} \rangle \delta^{\sigma'}{}_{\sigma} \sigma(s_0)^{s}{}_{s'} \end{aligned} \tag{2.18}$$

where we have used the fact that $(q_1 q_2)$ and $(q_3 q_4)$ are such that $\mathrm{S}(q_1 q_2)_{q_1} = \mathrm{S}(q_3 q_4)_{q_3} \equiv \mathrm{S}$ and $s_0 \in \mathrm{S}$. We incorporate this result into (2.17) and use (2.11) to combine the four induction coefficients replacing them with a 'lower' reinduction factor. The final result is

$$\begin{aligned} \sum_{a\eta b\gamma a_2} & \langle \lambda(\mathrm{L}) \uparrow c\kappa(\mathrm{K}) \uparrow d\gamma(\mathrm{G}) | \lambda(\mathrm{L}) \uparrow a\eta(\mathrm{H}) \uparrow b\gamma(\mathrm{G}) \rangle \\ & \times \langle \eta(\mathrm{H}) \uparrow b\gamma(\mathrm{G}) a_1 \mu(\mathrm{M}) | [\eta(\mathrm{H}_{q_1}) a_2 \nu(\mathrm{N}(q_1))] \uparrow b\mu(\mathrm{M}) \rangle \\ & \times \langle \lambda(\mathrm{L}) \uparrow a\eta(\mathrm{H}) a_2 \nu(\mathrm{N}(q_1)_{q_1^{-1}}) | [\lambda(\mathrm{L}_{q_2}) a_4' \sigma(\mathrm{S}(q_1 q_2))] \uparrow a'\nu(\mathrm{N}(q_1)_{q_1^{-1}}) \rangle \\ = \sum_{a_3 a_4 \rho c' d'} & \langle \kappa(\mathrm{K}) \uparrow d\gamma(\mathrm{G}) a_1 \mu(\mathrm{M}) | [\kappa(\mathrm{K}_{q_3}) a_3 \rho(\mathrm{R}(q_3))] \uparrow d'\mu(\mathrm{M}) \rangle \\ & \times \langle \lambda(\mathrm{L}) \uparrow a\kappa(\mathrm{K}) a_3 \rho(\mathrm{R}(q_3)_{q_3^{-1}}) | [\lambda(\mathrm{L}_{q_4}) a_4 \sigma(\mathrm{S}(q_3 q_4))] \uparrow c'\rho(\mathrm{R}(q_3)_{q_3^{-1}}) \rangle \\ & \times \langle \lambda(\mathrm{L}_{q_3 q_4}) a_4 \sigma(\mathrm{S}(q_3 q_4)_{q_3}) | \lambda(\mathrm{L}_{q_1 q_2}) a_4' \sigma(\mathrm{S}(q_1 q_2)_{q_1}) \rangle \\ & \times \langle \sigma(\mathrm{S}(q_3 q_4)_{q_3}) \uparrow c'\rho(\mathrm{R}(q_3)) \uparrow d'\mu(\mathrm{M}) | \\ & \times \sigma(\mathrm{S}(q_1 q_2)_{q_1}) \uparrow a'\nu(\mathrm{N}(q_1)) \uparrow b'\mu(\mathrm{M}) \rangle . \end{aligned} \tag{2.19}$$

We note that the last term—the reinduction factor—is very much dependent on the double coset representatives q_1, q_2, q_3, q_4 and that $\mathrm{S}(q_3, q_4)_{q_3} = \mathrm{S}(q_1, q_2)_{q_1}$.

3. An analytic choice for induction coefficients and its consequences

Sullivan (1973, 1975) has investigated the properties of double coset matrices. In this section we review this work although we cast the results into a form more suited for our purpose. We show that a particular choice of solution for the induction coefficients based on the double coset matrix element (DCME) leads to identifications between our induction factor and Sullivan's weighted DCME and between the reinduction factor and the resubduction factor.

The double coset matrix is the irreducible matrix representation of the double coset representative $q \in \mathrm{K}\backslash\mathrm{G}/\mathrm{H}$. An element of this matrix, the DCME, is labelled by indices symmetry adapted to the GH chains $\mathrm{G} \supset \mathrm{H} \supset \mathrm{L}(q)_{q^{-1}}$ and $\mathrm{G} \supset \mathrm{K} \supset \mathrm{L}(q)$. The DCME has the form (cf Sullivan 1975, equation (1.1))

$$\langle \gamma(\mathrm{G}) b\kappa(\mathrm{K}) d\lambda(\mathrm{L}(q)) l | q | \gamma(\mathrm{G}) a\eta(\mathrm{H}) c\lambda'(\mathrm{L}(q)_{q^{-1}}) l' \rangle$$
$$= \gamma(\mathrm{G}) b\kappa(\mathrm{K}) d\lambda(\mathrm{L}(q)) | \gamma(\mathrm{G}_q) a\eta(\mathrm{H}_q) c\lambda(\mathrm{L}(q)) \rangle \delta^{\lambda}{}_{\lambda'}, \delta^{l}{}_{l'}. \tag{3.1}$$

The weighted DCME given by

$$|\gamma\lambda\,\mathrm{HK}/\mathrm{GL}\eta\kappa|^{1/2} \langle \gamma(\mathrm{G}) b\kappa(\mathrm{K}) d\lambda(\mathrm{L}(q)) | \gamma(\mathrm{G}_q) a\eta(\mathrm{H}_q) c\lambda(\mathrm{L}(q)) \rangle \tag{3.2}$$

was shown to be orthonormal on the sets of indices $(a\gamma b)$ and $(c\lambda(\mathrm{L}(q))d)$ and considered to be the transformation between the representations equivalent by Mackey's subgroup theorem, see (2.5). Furthermore if $\mathrm{K} = \mathrm{E}$ then q corresponds to a coset decomposition G/H and (3.2) reduces to the form, which we call a weighted coset matrix element (weighted CME),

$$|\gamma\mathrm{H}/\mathrm{G}\eta|^{1/2} \langle \gamma(\mathrm{G}) i O(\mathrm{E}) | \gamma(\mathrm{G}_p) a\eta(\mathrm{H}_p) j O(\mathrm{E}) \rangle$$
$$= |\gamma\mathrm{H}/\mathrm{G}\eta|^{1/2} \langle \gamma(\mathrm{G}) i | p | \gamma(\mathrm{G}) a\eta(\mathrm{H}) j \rangle, \tag{3.3}$$

on replacing q by p, b by i and c by j. These remarks are to be compared to those for the induction factor. We also note that the CME is labelled by index sets belonging to two different basis labels. However by a suitable transformation the CME may be expressed in two forms

$$\langle \gamma(\mathrm{G}) i | p | \gamma(\mathrm{G}) a\eta(\mathrm{H}) j \rangle = \sum_{a'} \gamma(p)^{i}{}_{i'} \langle \gamma(\mathrm{G}) i' | \gamma(\mathrm{G}) a\eta(\mathrm{H}) j \rangle$$
$$= \sum_{a'\eta j'} \langle \gamma(\mathrm{G}) i | \gamma(\mathrm{G}) a'\eta'(\mathrm{H}) j' \rangle \gamma(p)^{a'\eta' j'}{}_{a\eta j} \tag{3.4}$$

where $\gamma(p)^{i}{}_{i'}$ and $\gamma(p)^{a'\eta' j'}{}_{a\eta j}$ are the irreducible matrix elements of the coset representative p in an arbitrary basis and a symmetry adapted basis respectively. The transformation coefficient $\langle \gamma(\mathrm{G}) i | \gamma(\mathrm{G}) a\eta(\mathrm{H}) j \rangle$ reduces the irrep space of $\gamma(\mathrm{G})$ under the chain $\mathrm{G} \supset \mathrm{H}$.

The comparison between the induction factor and weighted DCME brings us to consider the identification

$$\langle \eta(\mathrm{H}) \uparrow a\gamma(\mathrm{G}) i | \eta(\mathrm{H}) \uparrow \mathrm{G} p j \rangle = |\gamma\mathrm{H}/\mathrm{G}\eta|^{1/2} \langle \gamma(\mathrm{G}) i | p | \gamma(\mathrm{G}) a\eta(\mathrm{H}) j \rangle. \tag{3.5}$$

The weighted CME has the same orthonormality conditions as the induction coefficient as can be seen from the remark following (3.2) with K set to E. Furthermore (3.5) satisfies (2.4)—that is, it performs the reduction of the induced representation $\eta(\mathrm{H}) \uparrow \mathrm{G}$

(omitting group labels for the moment)

$$\sum_{p'j'pj} |\gamma H/G\eta| \langle \gamma' i' | p' | \gamma' a' nj' \rangle \delta^{p'}{}_{p_0} \eta(p_0^{-1} gp)^{j'}{}_j \langle \gamma a \eta j | p^{-1} | i \rangle$$

$$= \sum_{pj} |\gamma H/G\eta| \langle \gamma' i' | gp | \gamma' a' \eta j \rangle \langle \gamma a \eta j | p^{-1} | \gamma i \rangle$$

$$= \delta^a{}_{a'} \delta^\gamma{}_{\gamma'} \gamma(g)^{i'}{}_i . \tag{3.6}$$

The first line is obtained by the group property of irrep matrices and the second by the orthonormality condition.

The identification (3.5) amounts to a choice for the induction coefficient. The effect of this choice is to fix entirely the phase freedom (see Haase and Butler 1985) within the induced representation theory. This can be seen as follows. Let us restrict ourselves to transformations which leave the irrep matrices of $\gamma(G)$ and $\eta(H)$ invariant, i.e. we have a set of standard matrices which we want to remain as standard. The allowed set of transformations determine the phase freedom of the induction coefficient and the CME (and other coefficients). This may be written

$$\begin{aligned} \langle \eta(H) \uparrow \hat{a}\gamma(G) i | \eta(H) \uparrow Gpj \rangle &= \sum_a U(\eta\uparrow, \gamma)^{\hat{a}}{}_a \langle \eta(H) \uparrow a\gamma(G) i | \eta(H) \uparrow Gpj \rangle \\ \langle \gamma(G) i | p | \gamma(G) \hat{a} \eta(H) j \rangle &= \sum_a \langle \gamma(G) i | p | \gamma(G) a \eta(H) j \rangle U(\gamma, \eta)^{-1a}{}_{\hat{a}} \end{aligned} \tag{3.7}$$

where $U(\eta\uparrow, \gamma)$ and $U(\gamma, \eta)$ are termed the phase freedom matrices. If the choice (3.5) is made and is also to remain invariant under further possible phase freedom choices, we must impose the restriction

$$U(\eta\uparrow, \gamma)^{\hat{a}}{}_a = U(\gamma, \eta)^{-1a}{}_{\hat{a}} . \tag{3.8}$$

Note this restriction reflects the statement of the Frobenius reciprocity theorem (2.2). It also implies that no phase freedom remains within the induced representation theory, i.e. for the induction factor and reinduction factor. These must be determined in some manner from (3.5). Indeed the expressions (2.7) and (2.11) provide us with their solution. Using the orthonormality of the various transformation coefficients, (2.7) is rearranged as

$$\langle \eta(H) \uparrow a\gamma(G) b\kappa(K) | [\eta(H_q) c\lambda(L(q))] \uparrow d\kappa(K) \rangle \delta^{\lambda'}{}_\lambda \delta^{l'}{}_l$$

$$= \sum_{ik} \langle \gamma(G) b\kappa(K) k | \gamma(G) i \rangle \langle \eta(H) \uparrow a\gamma(G) i | \eta(H) \uparrow Gpj \rangle$$

$$\times \langle \lambda'(L(q)) \uparrow Krl' | \lambda'(L(q)) \uparrow d\kappa(K) k \rangle \langle \eta(H) j | \eta(H) c\lambda(L(q)_{q^{-1}}) l \rangle \tag{3.9}$$

$$= \sum_{ijk} \langle \gamma(G) b\kappa(K) k | \gamma(G) i \rangle \langle \gamma(G) i | p | \gamma(G) a \eta(H) j \rangle$$

$$\times |\gamma H/G\eta|^{1/2} |K\lambda/\kappa L|^{1/2} \langle \kappa(K) d\lambda'(L(q)) l' | r^{-1} | \kappa(K) k \rangle$$

$$\times \langle \eta(H) j | \eta(H) c\lambda(L(q)_{q^{-1}}) l \rangle \tag{3.10}$$

$$= |\gamma\lambda HK/GL\eta\kappa|^{1/2} \langle \gamma(G) b\kappa(K) d\lambda'(L(q)) l' | r^{-1} p | \gamma(G) a\eta(H) c\lambda(L(q)_{q^{-1}}) l \rangle \tag{3.11}$$

where we have used the group property. The final result is

$$\langle \eta(H) \uparrow a\gamma(G) b\kappa(K) | [\eta(H_q) c\lambda(L(q))] \uparrow d\kappa(K) \rangle$$

$$= |\gamma\lambda HK/GL\eta\kappa|^{1/2} \langle \gamma(G) b\kappa(K) d\lambda(L(q)) | \gamma(G_q) a\eta(H_q) c\lambda(L(q)) \rangle \tag{3.12}$$

where we have used (3.1). The right-hand side is Sullivan's weighted DCME of (3.2). The reinduction factor is evaluated to be a resubduction factor (see Haase and Butler 1984a, equation (2.23))

$$\langle\lambda(\mathrm{L})\uparrow c\kappa(\mathrm{K})\uparrow d\gamma(\mathrm{G})\,|\,\lambda(\mathrm{L})\uparrow a\eta(\mathrm{H})\uparrow b\gamma(\mathrm{G})\rangle$$

$$= \sum_{ijkll'} |\gamma\mathrm{L}/\mathrm{G}\lambda|\,|\gamma|^{-1}\langle\gamma(\mathrm{G})i|p_3|\gamma(\mathrm{G})d\kappa(\mathrm{K})k\rangle\langle\kappa(\mathrm{K})k|p_4|\kappa(\mathrm{K})c\lambda(\mathrm{L})l'\rangle$$

$$\times\langle\lambda(\mathrm{L})l'|l_0|\lambda(\mathrm{L})l\rangle\langle\eta(\mathrm{H})a\lambda(\mathrm{L})l|p_2^{-1}|\eta(\mathrm{H})j\rangle\langle\gamma(\mathrm{G})b\eta(\mathrm{H})j|p_1^{-1}|\gamma(\mathrm{G})i\rangle \tag{3.13}$$

$$= \sum_{l} |\mathrm{L}/\mathrm{G}|\,|\lambda|^{-1}\langle\gamma(\mathrm{G})b\eta(\mathrm{H})a\lambda(\mathrm{L})l|p_3p_4l_0p_2^{-1}p_1^{-1}|\gamma(\mathrm{G})d\kappa(\mathrm{K})c\lambda(\mathrm{L})l\rangle \tag{3.14}$$

$$= \langle\gamma(\mathrm{G})b\eta(\mathrm{H})a\lambda(\mathrm{L})\,|\,\gamma(\mathrm{G})d\kappa(\mathrm{K})c\lambda(\mathrm{L})\rangle. \tag{3.15}$$

To initiate this proof we have used the orthonormality of the induction coefficients to remove them to the RHS of (2.11), and later the group property of irrep matrices. The summation over all p_i is restricted such that $l_0 = p_4^{-1}p_3^{-1}p_1p_2 \in \mathrm{L}$. Finally the summation over p_1 and p_2 enumerates the number of cosets for G/L which is $|\mathrm{G}/\mathrm{L}|$. Note that both induction and subduction factors are indexed by the same labels $(c\kappa d)$ and $(a\eta b)$—again a consequence of the Frobenius reciprocity theorem.

One can also show that (3.12) and (3.15) satisfies identity (2.19). The proof will not be given here since it is tedious and lengthy, and does not add further to the discussion.

A final consequence of the choice (3.5) is that the overlap $\langle\eta\uparrow p'j'\,|\,\eta\uparrow pj\rangle$ (also known as the normalisation kernel in the collective theory of nuclei) is not necessarily orthonormal. We have upon expanding in terms of the reduced basis

$$\langle\eta\uparrow p'j'\,|\,\eta\uparrow pj\rangle$$

$$= \sum_{\alpha\gamma i}\langle\eta\uparrow p'j'\,|\,\eta\uparrow a\gamma i\rangle\langle\eta\uparrow a\gamma i\,|\,\eta\uparrow pj\rangle$$

$$= \left|\frac{\gamma}{\mathrm{G}}\;\frac{\mathrm{H}}{\eta}\right|\langle\gamma a\eta j'|p'^{-1}|\gamma i\rangle\langle\gamma i|p|\gamma a\eta j\rangle$$

by (3.5),

$$= \sum_{\gamma a}\left|\frac{\gamma}{\mathrm{G}}\;\frac{\mathrm{H}}{\eta}\right|^{1/2}\gamma(p'^{-1}p)^{a\eta j'}{}_{a\eta j} \tag{3.16}$$

by the group matrix property.

If $p'^{-1}p \in \mathrm{H}$ this reduces to

$$\langle\eta\uparrow p'j'\,|\,\eta\uparrow pj\rangle = \eta(p'^{-1}p)^{j'}{}_{j} \tag{3.17}$$

and if $p'^{-1}p = e$ the overlap becomes orthonormal

$$\langle\eta\uparrow pj'\,|\,\eta\uparrow pj\rangle = \delta^{j'}{}_{j}. \tag{3.18}$$

As an example of the use of (3.5) we give an application to the group-subgroup chain $\mathrm{SO}(n+1)\supset\mathrm{SO}(n)$. First though we use (3.4) to rearrange (3.5) into the form

$$\langle \eta(\mathrm{H}) \uparrow a\gamma(\mathrm{G})a'\eta'(\mathrm{H})j' | \eta(\mathrm{H}) \uparrow \mathrm{G}pj\rangle$$
$$= \left|\frac{\gamma}{\mathrm{G}}\frac{\mathrm{H}}{\eta}\right|^{1/2} \langle \gamma(\mathrm{G})a'\eta'(\mathrm{H})j'|p|\gamma(\mathrm{G})a\eta(\mathrm{H})j\rangle$$
$$= \left|\frac{\gamma}{\mathrm{G}}\frac{\mathrm{H}}{\eta}\right|^{1/2} \gamma(p)^{a\eta' j'}{}_{a\eta j} \tag{3.19}$$

where the CME is now expressed in the one GH adapted basis. The coset space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ can be identified with the n-dimensional sphere S_n embedded in $\mathrm{SO}(n+1)$. The volume of this space is

$$|\mathrm{SO}(n+1)/\mathrm{SO}(n)| = \frac{2^{\delta}(2\pi)^d}{(n-1)(n-3)\ldots} \tag{3.20}$$

where $\delta = (n+1) \bmod 2$ and d is the largest integer $\leq \frac{1}{2}(n+1)$. The coset representatives are parametrised by n variables x_i $(i = 1 \ldots n)$. The range of the x_i is governed by $\Sigma_{i=1}^{n} x_i^2 \leq 1$. Setting $x_{n+1} = \pm(1 - \Sigma_i x_i^2)^{1/2}$, the variables then obey $\Sigma_{i=1}^{n+1} x_i^2 = 1$, i.e. the S_n constraint. For the fundamental representation, the coset representatives are

$$D^{[1]}(p) = D^{[1]}(x_1 \ldots x_n) = \begin{pmatrix} [I_n - \boldsymbol{x} \cdot \boldsymbol{x}^{\mathrm{t}}]^{1/2} & \boldsymbol{x} \\ -\boldsymbol{x}^{\mathrm{t}} & [1 - \boldsymbol{x}^{\mathrm{t}}\boldsymbol{x}]^{1/2} \end{pmatrix} \tag{3.21}$$

where $\boldsymbol{x} \cdot \boldsymbol{x}^{\mathrm{t}}$ and $\boldsymbol{x}^{\mathrm{t}} \cdot \boldsymbol{x}$ are understood to be the matrix products of column matrix $\boldsymbol{x}$ and row matrix $\boldsymbol{x}^{\mathrm{t}}$, and I_n is the $n \times n$ unit matrix. In terms of the projective coordinate parameters $z_i = [2/(1 + x_{n+1})]x_i$ the coset representatives are

$$D^{[1]}(p) = D^{[1]}(z_1 \ldots z_n) = \begin{pmatrix} \left[I_n - \dfrac{\boldsymbol{z} \cdot \boldsymbol{z}^{\mathrm{t}}}{(1 + z^2/4)^2}\right] & \dfrac{\boldsymbol{z}}{1 + z^2/4} \\ \dfrac{-\boldsymbol{z}^{\mathrm{t}}}{1 + z^2/4} & \dfrac{1 - z^2/4}{1 + z^2/4} \end{pmatrix} \tag{3.22}$$

using $x_{n+1} = [(1 - z^2/4)/1 + z^2/4]$ and $z^2 = \Sigma\, z_i z_i$. Both $D^{[1]}(x)$ and $D^{[1]}(z)$ are symmetry adapted to the $\mathrm{SO}(n+1) \supset \mathrm{SO}(n)$ chain.

We now consider the permutation representation, i.e. the representation induced from the identity irrep $[O](\mathrm{SO}(n))$. This representation is an infinite-dimensional representation of $\mathrm{SO}(n+1)$ which under reduction to the irreps of $\mathrm{SO}(n+1)$ gives all the total symmetric irreps $[l]$ of $\mathrm{SO}(n+1)$ with multiplicity one. This follows from the Frobenius reciprocity theorem. Thus (3.16) reduces to

$$\langle [O](\mathrm{SO}_n) \uparrow o[l](\mathrm{SO}_{n+1})o[m](\mathrm{SO}_n)j | [O](\mathrm{SO}_n) \uparrow \mathrm{SO}_{n+1}po\rangle$$
$$= |[l]/\mathrm{G}/\mathrm{H}|^{1/2} D^{[l]}(p)^{o[m]j}{}_{o[o]o} \tag{3.23}$$

where

$$|[l]/\mathrm{G}/\mathrm{H}|^{1/2} = \left((n + 2l - 2)\frac{(n+l-3)!}{l!(n-2)!}\frac{(n-1)(n-3)\ldots}{2^{\delta}(2\pi)^d}\right)^{1/2}.$$

The CME for $[l]$ $(l > 1)$ can be obtained by successive couplings of $D^{[1]}(p)$ if the coupling coefficients are known.

In the case $n = 2$, i.e. $\mathrm{SO}(3) \supset \mathrm{SO}(2)$, the right-hand side is found to be $[(2l + 1)/\pi]^{1/2} D^{[l]}(p)^{o[m]o}{}_{o[o]o}$ which is noted as being the well known spherical harmonics $Y_{lm}(\alpha\beta)$ if the parametrisation of p is taken from the Euler angles (α, β, γ) of $\mathrm{SO}(3)$.

4. Concluding remarks

In this paper we have given the algebra of the induced representation theory, specifically the identities relating the three transformation coefficients of this theory—induction coefficients, induction factors and reinduction factors. The weighted CME was found to satisfy the identities of the induction coefficient. The identification was shown to fix the induced representation algebra which we summarise below:

induction coefficient = weight CME
induction factor = weight DCME
reinduction factor = resubduction factor.

This last result has importance for the Racah-Wigner algebra of the unitary groups. From the Schur-Weyl duality (see Haase and Butler 1984b) we have the correspondences

(i) $U_{n_1+n_2+n_3}$ resubduction factor $\sim S_{f_1+f_2+f_3}$ reinduction factor.
(ii) U_n recoupling factor $\sim S_{f_1+f_2+f_3}$ resubduction factor.

Thus the identification of the symmetric group resubduction and reinduction factors implies an equivalence between the U_n recoupling factor and the $U_{n_1+n_2+n_3}$ resubduction factor to within certain duality factors.

References

Barut A O and Raçska R 1977 *Theory of Group Representations and Applications* (Warsaw: Polish Science)
Bradley C J and Cracknell A P 1972 *The Mathematical Theory of Symmetry in Solids* (Oxford: Clarendon)
Butler P H 1975 *Phil. Trans. R. Soc.* **277** 545-85
Chan K S and Newman D J 1984 *J. Phys. A: Math. Gen.* **17** 253-65
Coleman A J 1966 *Queen's Papers in Pure and Appl. Math. No 4* Queen's University, Kingston, Ontario
—— 1975 *Group Theory and its Applications* ed E M Loebl (New York: Academic) pp 57-118
Frobenius G 1898 *Sitz Preuss. Akad. Wiss. Berlin* 501-15
Haase R W and Butler P H 1984a *J. Phys. A: Math. Gen.* **17** 47-59
—— 1984b *J. Phys. A: Math. Gen.* **17** 61-74
—— 1986 *J. Phys. A: Math. Gen.* **19** 1065-82
Kramer P 1982 *Ann. Phys., NY* **141** 269-82
Mackey G W 1951 *Am. J. Phys.* **73** 576-92
—— 1952 *Ann. Math.* **55** 101-39
—— 1953 *Ann. Math.* **58** 193-221
—— 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)
Monastryrskii M I and Perelomov A M 1975 *Ann. Inst. H Poincaré* **XXIII** 23-48
Perelomov A M 1975 *Commun. Math. Phys.* **44** 197-210
Rowe D J 1984 *Preprint, Coherent State Theory of the Non Compact Sympletic Group* Toronto University
Salam A and Strathdee J 1982 *Ann. Phys., NY* **141** 316-52
Sullivan J J 1973 *J. Math. Phys.* **14** 387-95
—— 1975 *J. Math. Phys.* **16** 1707-9
van Nieuwenhuizen P 1984 *Supersymmetry and Supergravity. Proc. Trieste Spring School, 4-14 April 1984* ed B de Wit, P Fayet and P van Nieuwenhuizen (Singapore: World Scientific) pp 239-323
Wigner E P 1939 *Ann. Math.* **40** 199-204